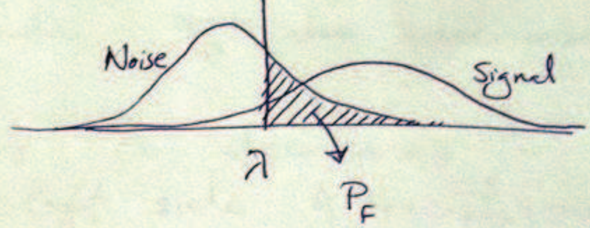
Ele 629 - Problem Set 1
Solutions

Problem (1):

a) $P_F = P_r\{X_n > \lambda\}$

$= 1 - P_r\{X_n \leq \lambda\}$

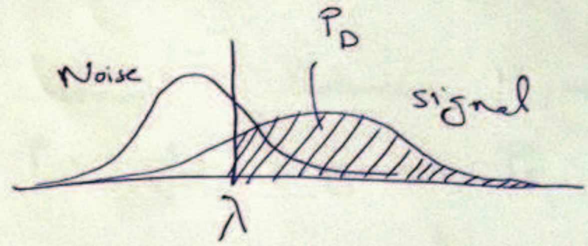
$= 1 - F_n(\lambda)$



b) $P_D = P_r\{X_s > \lambda\}$

$= 1 - P_r\{X_s \leq \lambda\}$

$= 1 - F_s(\lambda)$



c) $X_n \sim N(\mu_n, \sigma_n^2)$, $X_s \sim N(\mu_s, \sigma_s^2)$

$$N(\mu, \sigma) \Rightarrow f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\therefore P_F = P_r\{X_n > \lambda\}$$

$$= \int_\lambda^\infty \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(x-\mu_n)^2}{2\sigma_n^2}} dx$$

let $y = \frac{x - \mu_n}{\sigma}$ $\qquad dy = \frac{dx}{\sigma_n}$

$$\Rightarrow P_F = \int_{\frac{\lambda-\mu_n}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$\boxed{P_F = Q\left(\frac{\lambda - \mu_n}{\sigma_n}\right)}$$

Similarly

$$\boxed{P_D = Q\left(\frac{\lambda - \mu_s}{\sigma_s}\right)}$$

d) $\lambda = 1.2$ $\qquad X_n \sim N(0, 1)$ $\qquad X_s \sim N(1.5, 4)$

$P_F = Q\left(\frac{1.2 - 0}{1}\right) = Q(1.2) = 0.11507$ (from Table)

$P_D = Q\left(\frac{1.2 - 1.5}{2}\right) = Q(-0.15) =$

e) If λ increases (i.e. moves to the right of the figure).

P_F decreases while P_D also decrease

On the other hand, as λ decreases (i.e. moves to the left side of the figure)

Both P_F & P_D increase

However, there exist a certain λ between μ_n, and μ_s that acheives the max P_D for a given P_F.

Problem (2):

a) $P_F = P_r(Y[0] \mid H_0)$

$$= \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= Q(\lambda)$$

$$P_f = \alpha \Rightarrow \lambda = Q^{-1}(\alpha)$$

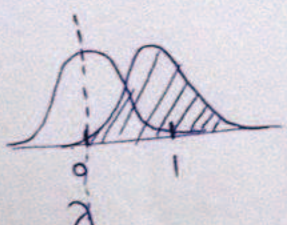
b) $P_D = P_r(Y[0] \mid H_1)$

$$= \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx$$

$$= Q(\lambda - 1) = Q(Q^{-1}(\alpha) - 1)$$

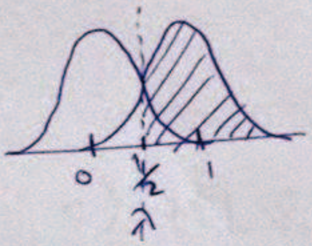
c) * $\alpha = 0.5 \xrightarrow{\text{table}} \lambda = Q^{-1}(0.5) = 0$

$$P_D = Q(0-1) = Q(-1) = 1 - Q(1)$$
$$= 1 - 0.15866$$
$$= 0.84134$$



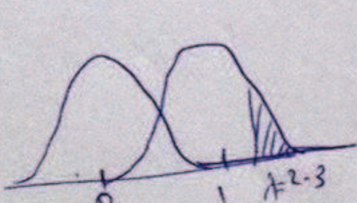
* $\alpha = 0.3 \xrightarrow{\text{table}} \lambda = Q^{-1}(0.3) \cong 0.5$

$$P_D = Q(0.5 - 1) = P(-0.5) = 1 - Q(0.5)$$
$$= 1 - 0.3$$
$$= 0.7$$



* $\alpha = 0.01 \xrightarrow{\text{table}} \lambda = Q^{-1}(0.01) \cong 2.3$

$$P_D = Q(2.3 - 1) = P(1.3) = 0.0968D$$



As P_F decrease from 0.5, we can achieve $P_D > 0.5$ as long as λ is greater than 0 & less than 1. If λ is further increased P_D will be less than 1/2 which is not

Problem (3): $\mathcal{H}_1 : \; Y[n] \sim \mathcal{N}(0, \sigma_0^2)$

$\mathcal{H}_0 : \; Y[n] \sim \mathcal{N}(0, \sigma_1^2) \quad , \quad (\sigma_1^2 > \sigma_0^2)$

n are i.i.d.

a) Since the N samples are i.i.d., the joint denisty of $\underline{Y}$ is simply the product of the indvidual probalilities

$$P(\underline{Y} \mid H_0) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_0^2}} \, e^{-\frac{(Y[n])^2}{2\sigma_0^2}}$$

&

$$P(\underline{Y} \mid H_1) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_1^2}} \, e^{-\frac{(Y[n])^2}{2\sigma_1^2}}$$

b) Maximum likelihood ratio

$$\frac{P(\underline{Y} \mid H_1)}{P(\underline{Y} \mid H_0)} = \frac{\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_1^2}} \, e^{-\frac{(Y[n])^2}{2\sigma_1^2}}}{\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_0^2}} \, e^{-\frac{(Y[n])^2}{2\sigma_0^2}}}$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^N \frac{e^{-\frac{1}{2\sigma_1^2}\left[\sum_{n=1}^{N} (Y[n])^2\right]}}{e^{-\frac{1}{2\sigma_0^2}\left[\sum_{n=1}^{N} (Y[n])^2\right]}}$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^N e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\left[\sum_{n=1}^{N} (Y[n])^2\right]}$$

c) MLT:

$$\frac{P(\underline{Y} \mid H_1)}{P(\underline{Y} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\left(\frac{\sigma_0}{\sigma_1}\right)^N e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\left[\sum_{n=1}^{N} (Y[n])^2\right]} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

d) Because the log function is monotonic and both sides are positive. we take take the log of the MLT without affecting the test

$$\log \frac{P(\underline{Y} \mid H_1)}{P(\underline{Y} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \gamma$$

$$N \ln \frac{\alpha_0}{\alpha_1} - \frac{1}{2}\left(\frac{1}{\alpha_1^2} - \frac{1}{\alpha_0^2}\right) \sum_{n=1}^{N} (Y[n])^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln \gamma$$

since $\frac{1}{\alpha_1^2} < \frac{1}{\alpha_0^2}$, we have

$$N \ln \frac{\alpha_0}{\alpha_1} + \frac{1}{2}\left(\frac{1}{\alpha_0^2} - \frac{1}{\alpha_1^2}\right) \sum_{n=1}^{N} (Y[n])^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln \gamma$$